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Picone's formula for linear non-selfadjoint impulsive differential equations

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Abstract

In this paper, we derive a Picone type formula for second-order linear non-selfadjoint impulsive differential equations having fixed moments of impulse actions, and obtain a Wirtinger type inequality, a Leighton type comparison theorem, and a Sturm–Picone comparison theorem for such equations. Moreover, several oscillation criteria are also derived as applications.

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1. Introduction

In the last few decades the theory of impulsive differential equations has been developed very rapidly due to the fact that such equations find a wide range of applications modeling adequately many real processes observed in physics, chemistry, biology, engineering, etc. [1,11,13,19–21,24–26]. Although numerous aspects of qualitative theory were also contained in the monographs [9,17], there appears to be less known about the oscillation theory, especially the Sturmian theory, of impulsive differential equations when compared to equations without impulses. Therefore, our objective is to make a contribution to the im-

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pulsive differential equations in this direction. Specifically, we are interested in a Picone's formula so as to obtain comparison theorems of Leighton and Sturm–Picone types for second-order linear impulsive differential equations with damping. Examples are provided to illustrate the importance of the results.

For our purpose, we fix $t_0 \in \mathbb{R}$ and let $\{\theta_i\}$ be a given strictly increasing sequence in $[t_0, \infty)$, $\theta_1 > t_0$. Let I be an interval contained in $[t_0, \infty)$. To simplify the statements in the theorems, we introduce the space of functions $\text{PLC}(I)$ as the set of functions $z : I \rightarrow \mathbb{R}$ which are continuous for $t \neq \theta_i$ and left continuous with discontinuities of the first kind at $t = \theta_i$. The space of functions $\text{PLC}^1(I)$ is defined to be the set of functions z such that $z, z' \in \text{PLC}(I)$. If $z \in \text{PLC}(I)$ then by $\Delta z|_{t=\theta_i}$ we denote the jump at $t = \theta_i$, i.e., $\Delta z|_{t=\theta_i} = z(\theta_i^+) - z(\theta_i)$, where $z(\theta_i^+) = \lim_{t \rightarrow \theta_i^+} z(t)$. Note that if $z \in \text{PLC}(I)$ and $\Delta z|_{t=\theta_i} = 0$ for all $i \in \mathbb{N}$, then z becomes continuous and conversely.

Consider the second-order linear impulsive differential equations of the form

$$\begin{aligned} l[x] &= (k(t)x')' + r(t)x' + p(t)x = 0, \quad t \neq \theta_i, \\ l_0[x] &= \Delta(k(t)x') + p_i x = 0, \quad t = \theta_i, \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} L[y] &= (m(t)y')' + s(t)y' + q(t)y = 0, \quad t \neq \theta_i, \\ L_0[y] &= \Delta(m(t)y') + q_i y = 0, \quad t = \theta_i, \end{aligned} \quad (1.2)$$

where $\{p_i\}$ and $\{q_i\}$ are real sequences and $k, m, r, s, p, q \in \text{PLC}(I)$ with $k(t) > 0$ and $m(t) > 0$ for all $t \in I$.

By a solution of (1.1) on an interval $I \subset [t_0, \infty)$ we mean a non-trivial continuous function $x(t)$ defined on I such that $x' \in \text{PLC}(I)$, $kx' \in \text{PLC}^1(I)$, and $x(t)$ satisfies (1.1). It is not difficult to see that such solutions exist.

The proof of the well-known Sturm–Picone comparison theorem (with $r(t) \equiv 0$, $s(t) \equiv 0$) [16] (see also [8,23]) is based on employing the Picone's formula

$$\begin{aligned} \frac{x}{y}(y k x' - x m y') \Big|_a^b &= \int_a^b \left[(k - m)(x')^2 + (q - p)x^2 + m \left(x' - \frac{x}{y} y' \right)^2 \right. \\ &\quad \left. + \frac{x}{y} \{ y l[x] - x L[y] \} \right] dt, \end{aligned} \quad (1.3)$$

which holds for all real-valued functions x and y defined on an interval $[a, b]$ such that x, y, kx' and my' are differentiable on $[a, b]$ and $y \neq 0$ for $t \in [a, b]$. The formula (1.3) has also been used for establishing Wirtinger type inequalities for solutions of ordinary differential equations [7,22], and generalized to second-order linear equations with damping terms [7, p. 11] and more recently to half-linear equations [6].

We observe that the first investigation of oscillatory properties of impulsive differential equations is due Gopalsamy and Zhang [4]. Later, several investigations have been done with respect to the oscillatory properties of various classes of impulsive differential equations, we may refer in particular to [2,3,5,12,14,18] and references cited therein. As far as the Sturmian theory is concerned, to the best of our knowledge, the first work has appeared in the literature in 1996, in which Bainov, Domshlak and Simeonov [2] studied the

Sturmian comparison theory for second-order linear impulsive differential equations of the form

$$x'' + p(t)x = 0, \quad t \neq \theta_i; \quad \Delta x' + p_i x = 0, \quad t = \theta_i.$$

Very recently, the present authors have developed a Sturmian theory for linear and half-linear impulsive differential equations [14].

The purpose of this study is to modify (1.3) and thereby extend the results in [7] to linear impulsive differential equations with damping and also generalize some of the results given in [2,14]. In particular, we establish a Wirtinger type inequality and a Leighton type comparison theorem together with some oscillation criteria for (1.1). As usual a solution is called oscillatory if it has arbitrarily large zeros, and a differential equation is oscillatory if every solution of the equation is oscillatory.

2. The main results

Let I_0 be a non-degenerate subinterval of I . In what follows we shall make use of the following condition:

$$k(t) \neq m(t) \quad \text{whenever } r(t) \neq s(t) \text{ for all } t \in I_0. \quad (\text{H})$$

It is well known that condition (H) is crucial in obtaining a Picone's formula in the case when impulses are absent. If (H) fails to hold then Wirtinger, Leighton, and Sturm–Picone type results require employing a so called “device of Picard.” We will show how this is possible for impulsive differential equations as well.

Let (H) be satisfied. Suppose that x and y are continuous functions defined on I_0 such that $x', y' \in \text{PLC}(I_0)$ and $kx', my' \in \text{PLC}^1(I_0)$. These simply mean that x and y are in the domain of l, l_0 and L, L_0 , respectively. If $y(t) \neq 0$ for any $t \in I_0$, then we may define

$$w(t) = \frac{x(t)}{y(t)} [y(t)k(t)x'(t) - x(t)m(t)y'(t)] \quad \text{for } t \in I_0.$$

For clarity we suppress the variable t . Clearly,

$$\begin{aligned} w' &= (k-m)(x')^2 + (q-p)x^2 + m\left(x' - \frac{x}{y}y'\right)^2 + x^2 \frac{sy'}{y} - rx x' \\ &\quad + \frac{x}{y} \{y l[x] - x L[y]\}, \quad t \neq \theta_i, \end{aligned} \quad (2.1)$$

$$\Delta w = x \{l_0[x] - p_i x\} - \frac{x^2}{y} \{L_0[y] - q_i y\}, \quad t = \theta_i. \quad (2.2)$$

In view of (1.1) and (1.2) it is not difficult to see, cf. [7], from (2.1) and (2.2) that

$$\begin{aligned} w' &= (k-m)(x')^2 + (q-p)x^2 + m\left(x' - \frac{xy'}{y}\right)^2 - sx\left(x' - \frac{xy'}{y}\right) \\ &\quad + (s-r)xx' + \frac{x}{y} \{y l[x] - x L[y]\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right\} x^2 + (k-m) \left\{ x' + \frac{(s-r)}{2(k-m)} x \right\}^2 \\
&\quad + \frac{m}{y^2} \left(x'y - xy' - \frac{s}{2m} xy \right)^2 + \frac{x}{y} \{ y l[x] - x L[y] \}, \quad t \neq \theta_i,
\end{aligned} \tag{2.3}$$

and

$$\Delta w = (q_i - p_i)x^2 + \frac{x}{y} \{ y l_0[x] - x L_0[y] \}, \quad t = \theta_i. \tag{2.4}$$

Employing the identity

$$w(\beta) - w(\alpha) = \int_{\alpha}^{\beta} w'(t) dt + \sum_{\alpha \leq \theta_i < \beta} \Delta w(\theta_i),$$

we easily obtain the following Picone's formula.

Theorem 2.1 (Picone's formula). *Let (H) be satisfied. Suppose that x and y are continuous functions defined on I_0 such that $x', y' \in \text{PLC}(I_0)$ and $kx', my' \in \text{PLC}^1(I_0)$. If $y(t) \neq 0$ for any $t \in I_0$, and $[\alpha, \beta] \subseteq I_0$, then*

$$\begin{aligned}
&\frac{x}{y} (y k x' - x m y') \Big|_{\alpha}^{\beta} \\
&= \int_{\alpha}^{\beta} \left\{ \left[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right] x^2 \right. \\
&\quad + (k-m) \left\{ x' + \frac{(s-r)}{2(k-m)} x \right\}^2 + \frac{m}{y^2} \left(x'y - xy' - \frac{s}{2m} xy \right)^2 \\
&\quad \left. + \frac{x}{y} \{ y l[x] - x L[y] \} \right\} dt + \sum_{\alpha \leq \theta_i < \beta} \left[(q_i - p_i)x^2 + \frac{x}{y} \{ y l_0[x] - x L_0[y] \} \right].
\end{aligned} \tag{2.5}$$

In a similar manner we derive a Wirtinger type inequality.

Theorem 2.2 (Wirtinger type inequality). *If there exists a solution x of (1.1) such that $x \neq 0$ on (a, b) , then*

$$W[\eta] = \int_a^b \left\{ p\eta^2 - k \left(\eta' - \frac{r}{2k} \eta \right)^2 \right\} dt + \sum_{\alpha \leq \theta_i < b} p_i \eta^2 \leq 0, \quad \eta \in \Omega_{rk}, \tag{2.6}$$

where

$$\Omega_{rk} = \{ \eta \in C[a, b]: r\eta' \in \text{PLC}[a, b], k\eta' \in \text{PLC}^1[a, b], \eta(a) = \eta(b) = 0 \}.$$

Proof. Let x be a solution of (1.1) such that $x(t) \neq 0$ for any $t \in (a, b)$. Setting $m \equiv k$, $q \equiv p$, $s \equiv r$, and $q_i = p_i$, replacing x by η and y by x in (2.3) and (2.4), we see that

$$\begin{aligned} w' &= k \left(\eta' - \frac{\eta}{x} x' \right)^2 + \eta^2 \frac{rx'}{x} - r\eta\eta' + \eta l[\eta] \\ &= \eta(k\eta')' + \left(p - \frac{r^2}{4k} \right) \eta^2 + s\eta\eta' + \frac{k}{x^2} \left(\eta'x - \eta x' - \frac{r\eta x}{2k} \right)^2, \quad t \neq \theta_i, \end{aligned} \quad (2.7)$$

and

$$\Delta w = \eta \{ \Delta(k\eta') + p_i \eta \}, \quad t = \theta_i. \quad (2.8)$$

It is clear that if $x(a^+) \neq 0$ and $x(b^-) \neq 0$, then the last term in (2.7) is integrable over (a, b) . If $x(a^+) = 0$, then since $x'(a^+) \neq 0$ (otherwise, we have only the trivial solution) it follows that

$$\lim_{t \rightarrow a^+} \left\{ \frac{\eta'(t)x(t) - \eta(t)x'(t)}{x(t)} - \frac{r(t)\eta(t)}{2k(t)} \right\} = \eta'(a^+) - \eta'(a^+) - \frac{r(a^+)\eta(a^+)}{2k(a^+)} = 0.$$

The same argument applies if $x(b^-) = 0$. Thus, the last term in (2.7) is integrable on (a, b) .

We now claim that $w(a^+) = w(b^-) = 0$. Let us consider $w(a^+) = 0$. The case $w(b^-) = 0$ is similar. If $x(a^+) \neq 0$, then we certainly have $w(a^+) = 0$. In case $x(a^+) = 0$, it follows from

$$\lim_{t \rightarrow a^+} \frac{\eta(t)}{x(t)} = \lim_{t \rightarrow a^+} \frac{\eta'(t)}{x'(t)} < \infty$$

that

$$w(a^+) = \lim_{t \rightarrow a^+} \frac{\eta(t)}{x(t)} \{ k(t)\eta'(t)x(t) - k(t)\eta(t)x'(t) \} = 0.$$

Integrating (2.7) over (a, b) and using (2.8), we see that

$$\begin{aligned} &\int_a^b \eta(k\eta')' dt + \int_a^b \left\{ \left(p - \frac{r^2}{4k} \right) \eta^2 + r\eta\eta' \right\} dt \\ &+ \int_a^b \frac{k}{x^2} \left\{ \eta'x - \eta x' - \frac{r\eta x}{2k} \right\}^2 dt + \sum_{a \leq \theta_i < b} \eta \{ \Delta(k\eta') + p_i \eta \} = 0. \end{aligned}$$

Applying the integration by parts formula to the first integral leads to

$$W[\eta] = - \int_a^b \frac{k}{x^2} \left\{ \eta'x - \eta x' - \frac{r\eta x}{2k} \right\}^2 dt \leq 0. \quad \square$$

As a corollary we have the following criterion on the existence of a zero of a solution of (1.1). This result may be considered as an extension of Lemma 1.3 in [22] to impulsive equations.

Corollary 2.1. *If there exists an $\eta \in \Omega_{rk}$ such that $W[\eta] > 0$, then every solution x of (1.1) has a zero in (a, b) .*

As an immediate consequence of Corollary 2.1, we have the following oscillation result.

Corollary 2.2. *Suppose for any given $t_1 \geq t_0$ there exists an interval $(a, b) \subset [t_1, \infty)$ and a function $\eta \in \Omega_{rk}$ for which $W[\eta] > 0$, then (1.1) is oscillatory.*

Next, we provide a Leighton type comparison result between non-trivial solutions of (1.1) and (1.2), which may be considered as an extension of the classical comparison theorem of Leighton [10, Corollary 1].

Theorem 2.3 (Leighton type comparison). *Suppose that there exists a solution $x \in \Omega_{rk}$ of (1.1). If (H) is satisfied with $(a, b) \subset I_0$ and*

$$L[x] := \int_a^b \left\{ \left[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right] x^2 + (k-m) \left[x' + \frac{s-r}{2(k-m)} x \right]^2 \right\} dt \\ + \sum_{a \leq \theta_i < b} (q_i - p_i) x^2 > 0, \quad (2.9)$$

then every solution y of (1.2) must have at least one zero in (a, b) .

Proof. Let $\alpha = a + \epsilon$ and $\beta = b - \epsilon \in I_0$. Since x and y are solutions of (1.1) and (1.2) respectively, we have $l[x] \equiv l_0[x] \equiv L[y] \equiv L_0[y] \equiv 0$. Employing Picone's formula (2.5), we see that

$$\frac{x}{y} (y k x' - x m y') \Big|_{a+\epsilon}^{b-\epsilon} \\ = \int_{a+\epsilon}^{b-\epsilon} \left[\left\{ q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right\} x^2 + (k-m) \left\{ x' + \frac{(s-r)}{2(k-m)} x \right\}^2 \right. \\ \left. + \frac{m}{y^2} \left\{ x' y - x y' - \frac{s}{2m} x y \right\}^2 \right] dt + \sum_{a+\epsilon \leq \theta_i < b-\epsilon} (q_i - p_i) x^2. \quad (2.10)$$

As in the proof of Theorem 2.2, the functions under integral sign are all integrable and regardless of the values of $y(a)$ or $y(b)$, left-hand side of (2.10) tends to zero as $\epsilon \rightarrow 0^+$. Clearly (2.10) results in

$$L[x] \leq 0,$$

a contradiction to (2.9). \square

Corollary 2.3 (Sturm–Picone type comparison). *Let x be a solution of (1.1) having two consecutive zeros $a, b \in I_0$. Suppose (H) holds, and*

$$k \geq m, \quad (2.11)$$

$$q \geq p + \frac{(s-r)^2}{4(k-m)} + \frac{s^2}{4m} \quad (2.12)$$

for all $t \in [a, b]$, and

$$q_i \geq p_i \quad (2.13)$$

for all $i \in \mathbb{N}$ for which $\theta_i \in [a, b]$.

If either (2.11) or (2.12) is strict in a subinterval of $[a, b]$ or (2.13) is strict for some $i \in \mathbb{N}$, then every solution y of (1.2) must have at least one zero on (a, b) .

Remark. If there is no impulse, then Theorem 2.3 coincides with Theorem 2.1 in [7], and if $s(t) \equiv r(t) \equiv 0$, then Theorem 2.3 and Corollary 2.3 reduce to the results Theorems 2.1 and 2.2 in [14].

Corollary 2.4. *Suppose that conditions (2.11)–(2.12) are satisfied for all $t \in [t_*, \infty)$ for some integer $t_* \geq t_0$, and that (2.13) is satisfied for all $i \in \mathbb{N}$ for which $\theta_i \geq t_*$. If one of the inequalities (2.11)–(2.13) is strict, then (1.2) is oscillatory whenever any solution x of (1.1) is oscillatory.*

As a consequence of Theorem 2.3 and Corollary 2.3, we have the following oscillation result.

Corollary 2.5. *Suppose for any given $t_1 \geq t_0$ there exists an interval $(a, b) \subset [t_1, \infty)$ for which either the conditions of Theorem 2.3 or Corollary 2.3 are satisfied, then (1.2) is oscillatory.*

If (H) does not hold, we introduce a setting based on a device of Picard [15] (see also [7, p. 12]) that leads to different versions of Corollary 2.3. Indeed, for any $h \in \text{PLC}^1(I)$ we have

$$\frac{d}{dt}(x^2 h) = 2xx'h + x^2 h', \quad t \neq \theta_i.$$

Let

$$v := \frac{x}{y}(y k x' - x m y') + x^2 h, \quad t \in I.$$

It follows that

$$\begin{aligned} v' &= \left\{ q - p + h' - \frac{(s-r+2h)^2}{4(k-m)} - \frac{s^2}{4m} \right\} x^2 + (k-m) \left\{ x' + \frac{s-r+2h}{2(k-m)} x \right\}^2 \\ &\quad + \frac{m}{y^2} \left\{ x'y - xy' - \frac{s}{2m} xy \right\}^2, \quad t \neq \theta_i, \\ \Delta v &= (q_i - p_i) x^2 + x^2 \Delta h, \quad t = \theta_i. \end{aligned}$$

Assuming that $r, s \in \text{PLC}^1(I)$, the choice of $h = (r - s)/2$ yields

$$\begin{aligned} v' &= \left\{ q - p - \frac{s' - r'}{2} - \frac{s^2}{4m} \right\} x^2 + (k - m)(x')^2 \\ &\quad + \frac{m}{y^2} \left\{ x'y - xy' - \frac{s}{2m} xy \right\}^2, \quad t \neq \theta_i, \\ \Delta v &= \left\{ q_i - p_i - \frac{1}{2}(\Delta s - \Delta r) \right\} x^2, \quad t = \theta_i. \end{aligned}$$

Then, we have the following result.

Theorem 2.4 (A device of Picard). *Let $r, s \in \text{PLC}^1(I)$ and x be a solution of (1.1) having two consecutive zeros a and b in I . Suppose that*

$$k \geq m, \quad (2.14)$$

$$q \geq p + \frac{1}{2}(s' - r') + \frac{s^2}{4m} \quad (2.15)$$

for all $t \in [a, b]$, and that

$$q_i \geq p_i + \frac{1}{2}(\Delta s - \Delta r), \quad t = \theta_i, \quad (2.16)$$

for all $i \in \mathbb{N}$ for which $\theta_i \in [a, b]$.

If either (2.14) or (2.15) is strict in a subinterval of $[a, b]$ or (2.16) is strict for some i , then any solution y of (1.2) must have at least one zero in (a, b) .

Corollary 2.6. *Suppose that (2.14)–(2.15) are satisfied for all $t \in [t_*, \infty)$ for some integer $t_* \geq t_0$, and that (2.16) is satisfied for all $i \in \mathbb{N}$ for which $\theta_i \geq t_*$. If $r, s \in \text{PLC}^1[t_*, \infty)$ and one of the inequalities (2.14)–(2.16) is strict, then (1.2) is oscillatory whenever any solution x of (1.1) is oscillatory.*

As a consequence of Theorem 2.4, we have the following Leighton type comparison result which is analogous to Theorem 2.3.

Theorem 2.5 (Leighton type comparison). *Let $r, s \in \text{PLC}^1[a, b]$. If there exists a solution $x \in \Omega_{rk}$ of (1.1) such that*

$$\begin{aligned} \mathcal{L}[x] &:= \int_a^b \left\{ \left[q - p - \frac{1}{2}(s' - r') - \frac{s^2}{4m} \right] x^2 dt + (k - m)(x')^2 \right\} dt \\ &\quad + \sum_{a \leq \theta_i < b} \left\{ q_i - p_i - \frac{1}{2}(\Delta s - \Delta r) \right\} x^2 > 0, \end{aligned}$$

then every solution y of (1.2) must have at least one zero in (a, b) .

As a consequence of Theorems 2.4 and 2.5, we have the following oscillation result.

Corollary 2.7. Suppose for any given $t_1 \geq t_0$ there exists an interval $(a, b) \subset [t_1, \infty)$ for which either the conditions of Theorems 2.4 or 2.5 are satisfied, then (1.2) is oscillatory.

Moreover, it is possible to obtain results for (1.2) analogous to Theorem 2.2 and Corollary 2.1.

Theorem 2.6 (Wirtinger type inequality). If there exists a solution y of (1.2) such that $y \neq 0$ on (a, b) , then for $s \in \text{PLC}^1[a, b]$ and for all $\eta \in \Omega_{sm}$,

$$\mathcal{W}[\eta] := \int_a^b \left\{ \left(q - \frac{s^2}{2m} - \frac{s'}{2} \right) \eta^2 - m(\eta')^2 \right\} dt + \sum_{a \leq \theta_i < b} \left(q_i - \frac{1}{2} \Delta s \right) \eta^2 \leq 0.$$

Corollary 2.8. If there exists an $\eta \in \Omega_{sm}$ with $s \in \text{PLC}^1[a, b]$ such that $\mathcal{W}[\eta] > 0$, then every solution y of (1.2) must have at least one zero in (a, b) .

As an immediate consequence of Corollary 2.8, we have the following oscillation result.

Corollary 2.9. Suppose for any given $t_1 \geq t_0$ there exists an interval $(a, b) \subset [t_1, \infty)$ and a function $\eta \in \Omega_{sm}$ with $s \in \text{PLC}^1(I)$ for which $\mathcal{W}[\eta] > 0$, then (1.2) is oscillatory.

3. Applications

3.1. Oscillation of second-order linear impulsive equations with damping

In this section we provide some examples to illustrate the results obtained.

Example 3.1. Consider

$$\begin{aligned} x'' - 2\alpha x' + \alpha^2 x &= 0, & t \neq i, \\ \Delta x' + 2(1 + \coth \alpha)x &= 0, & t = i \ (i \in \mathbb{N}), \end{aligned} \quad (3.1)$$

where α is a fixed real number. It is easy to verify that $x(t) = x_i(t)$, where

$$x_i(t) = (-1)^i e^{\alpha(t-i)+1} \{ (e^\alpha + 1)(i - t) - 1 \}, \quad t \in (i - 1, i] \ (i \in \mathbb{N}),$$

is a solution of (3.1). Clearly, this solution is oscillatory with zeros at $t_i = i - (e^\alpha + 1)^{-1}$, $i \in \mathbb{N}$.

Remark. If the impulse conditions are dropped, then the equation has no oscillatory solution.

We may now apply Corollaries 2.4 and 2.6 to get the following oscillation criteria (a) and (b):

(a) If there exists an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} k(t) &\leq 1, \\ k(t) &< 1 \quad \text{whenever } r(t) \neq -2\alpha, \\ p(t) &\geq \alpha^2 + \frac{\{r(t) + 2\alpha\}^2}{4\{1 - k(t)\}} + \frac{r^2(t)}{4k(t)}, \\ p_i &\geq 2(1 + \coth \alpha) \end{aligned}$$

for all $t \geq n_0$, and for all $i \geq n_0$, then (1.1) with $\theta_i = i$ is oscillatory.

(b) If there exists an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} k(t) &\leq 1, \\ p(t) &\geq \alpha^2 + \frac{r'(t)}{2} + \frac{r^2(t)}{4k(t)}, \\ p_i &\geq 2(1 + \coth \alpha) + \frac{1}{2} \Delta r(i) \end{aligned}$$

for all $t \geq n_0$, and for all $i \geq n_0$, then (1.1) with $\theta_i = i$ is oscillatory.

The lemma below, cf. [2, Lemma 1] and [14, Lemma 3.1], provides more test equations for comparison purposes.

Lemma 3.1. *Let ψ be a positive and continuous function for $t \geq a$ with $\psi' \in \text{PLC}^1[a, \infty)$, where a is a fixed real number. Suppose that $k \in \text{PLC}^2[a, \infty)$ and $r \in \text{PLC}^1[a, \infty)$. Then the function*

$$x(t) = \frac{1}{\sqrt{k(t)\psi(t)}} \exp\left(-\frac{1}{2} \int_a^t \frac{r(s)}{k(s)} ds\right) \sin\left(\int_a^t \psi(s) ds\right), \quad t \geq a, \quad (3.2)$$

is a solution of

$$\begin{aligned} (k(t)x')' + r(t)x' + p(t)x &= 0, \quad t \neq \theta_i, \\ \Delta(k(t)x') + p_i x &= 0, \quad t = \theta_i \quad (i \in \mathbb{N}), \end{aligned}$$

where

$$\begin{aligned} p(t) &= \frac{1}{2} \left\{ k''(t) + r'(t) + r(t) \frac{k'(t)}{k(t)} + \frac{r^2(t)}{k(t)} \right\} - \frac{\{k'(t) + r(t)\}^2}{4k(t)} \\ &\quad + k(t) \left\{ \frac{\psi''(t)}{2\psi(t)} + \psi^2(t) - \frac{3}{4} \left(\frac{\psi'(t)}{\psi(t)} \right)^2 \right\}, \\ p_i &= \frac{1}{2\psi(\theta_i)} [\psi(\theta_i) \Delta k'(\theta_i) + k(\theta_i) \Delta \psi'(\theta_i)] + \frac{1}{2} \Delta r(\theta_i), \quad \theta_i > a. \end{aligned}$$

It is obvious that if

$$\int_a^\infty \psi(t) dt = \infty,$$

then $x(t)$ defined in (3.2) is oscillatory.

Clearly, Lemma 3.1 can be used to derive general oscillation criteria for (1.1). We prefer, however, to establish more concrete oscillation criteria by making use of the following particular cases of Lemma 3.1.

Example 3.2. Let $k(t) = t^2/4$, $r(t) = -t/4$ and $\psi(t) = \frac{2i-t}{i(i+1)}$, $i-1 < t \leq i$, $i \in \mathbb{N}$. In view of Lemma 3.1, we see that $x(t) = x_i(t)$, where

$$x_i(t) = \frac{2}{\sqrt{\beta t \psi(t)}} \sin \left(\int_1^t \psi(s) ds \right), \quad t \in (i-1, i] \quad (i \in \mathbb{N}),$$

is an oscillatory solution of

$$(t^2 x')' - t x' + \left\{ t^2 \left[\left(\frac{2i-t}{i(i+1)} \right)^2 - \frac{3}{4} \left(\frac{1}{2i-t} \right)^2 \right] - \frac{1}{4} \right\} x = 0, \quad t \in (i-1, i),$$

$$\Delta(t^2 x') + \frac{i}{i+2} x = 0, \quad t = i \quad (i \in \mathbb{N}).$$

Example 3.3. Let $k(t) = \xi_i^2(t+i)^2$, $r(t) = -\xi_i^2(t+i)$ and $\psi(t) = \frac{2i-t}{i(i+1)}$, $i-1 < t \leq i$, $i \in \mathbb{N}$, where

$$\xi_i = \frac{2^{2i-2}(i-1)!^2}{(2i-1)!}.$$

In view of Lemma 3.1, we see that $x(t) = x_i(t)$, where

$$x_i(t) = \frac{1}{\xi_i(t+i)\sqrt{\psi(t)}} \exp \left(\frac{1}{2} \int_{\gamma}^t \frac{ds}{s+i} \right) \sin \left(\int_1^t \psi(s) ds \right), \quad t \in (i-1, i],$$

is an oscillatory solution of

$$(\xi_i^2(t+i)^2 x')' - \xi_i^2(t+i) x' + \left\{ (t+i)^2 \left[\left(\frac{2i-t}{i(i+1)} \right)^2 - \frac{3}{4} \left(\frac{1}{2i-t} \right)^2 \right] - \frac{1}{4} \right\} \xi_i^2 x = 0, \quad t \in (i-1, i),$$

$$\Delta(\xi_i^2(t+i)^2 x') + \frac{i(7i+2)}{(i+2)(2i+1)} \xi_i^2 x = 0, \quad t = i \quad (i \in \mathbb{N}).$$

In view of the above examples, by applying Corollaries 2.4 and 2.6 we easily see that (1.1) with $\theta_i = i$ is oscillatory if there exists an $n_0 \in \mathbb{N}$ such that, for each fixed $i \geq n_0$ and for all $t \in (i-1, i]$, at least one of the following conditions (a)–(d) holds:

- (a) $k(t) \leq t^2$; $k(t) < t^2$ whenever $r(t) \neq -t$;
- $$p(t) \geq t^2 \left[\left(\frac{2i-t}{i(i+1)} \right)^2 - \frac{3}{4} \left(\frac{1}{2i-t} \right)^2 \right] - \frac{1}{4} + \frac{\{r(t)+t\}^2}{4\{t^2-k(t)\}} + \frac{r^2(t)}{4k(t)};$$
- $$p_i \geq \frac{i}{i+2}.$$

$$(b) \quad k(t) \leq t^2;$$

$$p(t) \geq t^2 \left[\left(\frac{2i-t}{i(i+1)} \right)^2 - \frac{3}{4} \left(\frac{1}{2i-t} \right)^2 \right] + \frac{1}{4} + \frac{r'(t)}{2} + \frac{r^2(t)}{4k(t)};$$

$$p_i \geq \frac{i}{i+2} + \frac{1}{2} \Delta r(i).$$

$$(c) \quad k(t) \leq \xi_i^2(t+i)^2; \quad k(t) < \xi_i^2(t+i)^2 \quad \text{whenever } r(t) \neq -\xi_i^2(t+i);$$

$$p(t) \geq \xi_i^2(t+i)^2 \left[\left(\frac{2i-t}{i(i+1)} \right)^2 - \frac{3}{4} \left(\frac{1}{2i-t} \right)^2 \right] - \frac{1}{4} \xi_i^2$$

$$+ \frac{\{r(t) + \xi_i^2(t+i)\}^2}{4\{\xi_i^2(t+i)^2 - k(t)\}} + \frac{r^2(t)}{4k(t)};$$

$$p_i \geq \frac{i(7i+2)}{(i+2)(2i+1)} \xi_i^2.$$

$$(d) \quad k(t) \leq \xi_i^2(t+i)^2;$$

$$p(t) \geq \xi_i^2(t+i)^2 \left[\left(\frac{2i-t}{i(i+1)} \right)^2 - \frac{3}{4} \left(\frac{1}{2i-t} \right)^2 \right] + \frac{1}{4} \xi_i^2 + \frac{r'(t)}{2} + \frac{r^2(t)}{4k(t)};$$

$$p_i \geq \frac{6i^2}{(i+2)(2i+1)} \xi_i^2 + \frac{1}{2} \Delta r(i).$$

3.2. Oscillation of second-order non-linear impulsive equations with damping

Consider the non-linear impulsive equations of the form

$$(m(t)z')' + s(t)z' + f(t, z, z') = 0, \quad t \neq \theta_i,$$

$$\Delta(m(t)z') + f_i(z, z') = 0, \quad t = \theta_i, \quad (3.3)$$

where $f(t, u, v)$ and $f_i(u, v)$, $i \in \mathbb{N}$, are real-valued continuous functions defined for all $t \geq t_0 \geq 0$ and for all $(u, v) \in \mathbb{R}^2$, m , s , and $\{\theta_i\}$ are as previously defined. It is tacitly assumed that there exist solutions of (3.3) which are continuous and defined for all $t \geq t_0$ satisfying $\sup\{|z(t)|, t \geq T\} > 0$ for all $T \geq t_0$. This last condition simply means that the solutions are non-trivial in the neighborhood of ∞ .

The following oscillation criteria can be easily established, cf. [2,14].

Theorem 3.1. Suppose that (H) holds, $k(t) \geq m(t)$, and

$$uf(t, u, v) \geq \left\{ p(t) + \frac{[s(t) - r(t)]^2}{4[k(t) - m(t)]} + \frac{s^2(t)}{4m(t)} \right\} u^2, \quad u f_i(u, v) \geq p_i u^2 \quad (3.4)$$

for all $t \geq t_0$ and for all $(u, v) \in \mathbb{R}^2$. If (1.1) is oscillatory, then so is (3.3).

Proof. Let us assume on the contrary that there exists a non-oscillatory solution $w(t)$ of (3.3) while every solution of (1.1) is oscillatory. Consider the linear impulsive system

$$\begin{aligned}(m(t)z')' + s(t)z' + q(t)z &= 0, \quad t \neq \theta_i, \\ \Delta(m(t)z') + q_i z &= 0, \quad t = \theta_i,\end{aligned}\tag{3.5}$$

where

$$q(t) = \frac{f(t, w(t), w'(t))}{w(t)}, \quad q_i = \frac{f_i(w(\theta_i), w'(\theta_i))}{w(\theta_i)}.$$

Clearly, $w(t)$ is also solution of (3.5). Let $x(t)$ be an oscillatory solution of (1.1) such that $x(a) = x(b) = 0$ and $x(t) > 0$ for all $t \in (a, b)$. Since $m(t) \leq k(t)$ by our hypothesis and

$$q(t) \geq p(t) + \frac{[s(t) - r(t)]^2}{4[k(t) - m(t)]} + \frac{s^2(t)}{4m(t)}$$

for $t \geq a$, and $q_i \geq p_i$ for all $i \in \mathbb{N}$ for which $\theta_i \geq a$ by (3.4), we may apply Corollary 2.3 to deduce that $w(t)$ must have a zero in (a, b) , which is a contradiction. \square

Alternatively, if (H) fails but $r, s \in \text{PLC}^1[t_0, \infty)$, then as an application of Theorem 2.4 we have the following result.

Theorem 3.2. *Suppose that $r, s \in \text{PLC}^1[t_0, \infty)$, $k(t) \geq m(t)$, and*

$$\begin{aligned}uf(t, u, v) &\geq \left\{ p(t) + \frac{1}{2}[s'(t) - r'(t)] + \frac{s^2(t)}{4m(t)} \right\} u^2, \\ uf_i(u, v) &\geq \left\{ p_i + \frac{1}{2}[\Delta s(\theta_i) - \Delta r(\theta_i)] \right\} u^2\end{aligned}$$

for all $t \geq t_0$ and for all $(u, v) \in \mathbb{R}^2$. If (1.1) is oscillatory, then so is (3.3).

References

- [1] J. Angelova, A. Dishliev, S. Nenov, I -Optimal curve for impulsive Lotka–Volterra predator–prey model, *Comput. Math. Appl.* 43 (2002) 1203–1218.
- [2] D.D. Bainov, Yu.I. Domshlak, P.S. Simeonov, Sturmian comparison theory for impulsive differential inequalities and equations, *Arch. Math.* 67 (1996) 35–49.
- [3] L. Berezansky, E. Braverman, Linearized oscillation theory for a nonlinear delay impulsive equation, *J. Comput. Appl. Math.* 161 (2003) 477–495.
- [4] K. Gopalsamy, B.G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl.* 139 (1989) 110–122.
- [5] Z. He, W. Ge, Oscillations of second-order nonlinear impulsive ordinary differential equations, *J. Comput. Appl. Math.* 158 (2003) 397–406.
- [6] J. Jaroš, T. Kusano, A Picone type identity for second order half-linear differential equations, *Acta Math. Univ. Comenian. (N.S.)* 1 (1999) 137–151.
- [7] K. Kreith, *Oscillation Theory*, Springer-Verlag, New York, 1973.
- [8] K. Kreith, Picone's identity and generalizations, *Rend. Mat.* 8 (1975) 251–262.
- [9] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, New Jersey, 1989.
- [10] W. Leighton, Comparison theorems for linear differential equations of second order, *Proc. Amer. Math. Soc.* 13 (1962) 603–610.

- [11] X. Liu, G. Ballinger, Boundedness for impulsive delay differential equations and applications to population growth models, *Nonlinear Anal.* 53 (2003) 1041–1062.
- [12] J. Luo, Second-order quasilinear oscillation with impulses, *Comput. Math. Appl.* 46 (2003) 279–291.
- [13] J.J. Nieto, Impulsive resonance periodic problems of first order, *Appl. Math. Lett.* 15 (2002) 489–493.
- [14] A. Özbekler, A. Zafer, Sturmian comparison theory for linear and half-linear impulsive differential equations, *Nonlinear Anal.* (2005), in press.
- [15] E. Picard, *Lecons Sur Quelques Problemes aux Limites de la Théorie des Equations Différentielles*, Paris, 1930.
- [16] M. Picone, Sui valori eccezionali di un parametro da cui dipende un equazione differenziale lineare ordinaria del second ordine, *Ann. Scuola Norm. Sup.* 11 (1909) 1–141.
- [17] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.
- [18] J. Shen, Qualitative properties of solutions of second-order linear ODE with impulses, *Math. Comput. Modelling* 40 (2004) 337–344.
- [19] J. Sun, Y. Zhang, Q. Wu, Less conservative conditions for asymptotic stability of impulsive control systems, *IEEE Trans. Automat. Control* 48 (2003) 829–831.
- [20] J. Sun, Y. Zhang, Impulsive control of Rössler systems, *Phys. Lett. A* 306 (2003) 306–312.
- [21] J. Sun, Y. Zhang, Impulsive control of a nuclear spin generator, *J. Comput. Appl. Math.* 157 (2003) 235–242.
- [22] C.A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.
- [23] C.A. Swanson, Picone’s identity, *Rend. Mat.* 8 (1975) 373–397.
- [24] S. Tang, L. Chen, Global attractivity in a “food-limited” population model with impulsive effects, *J. Math. Anal. Appl.* 292 (2004) 211–221.
- [25] Y.P. Tian, X. Yu, O.L. Chua, Time-delayed impulsive control of chaotic hybrid systems, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 14 (2004) 1091–1104.
- [26] S. Zhang, L. Dong, L. Chen, The study of predator–prey system with defensive ability of prey and impulsive perturbations on the predator, *Chaos Solitons Fractals* 23 (2005) 631–643.